

# MBR—A COMPUTER PROGRAM FOR PERFORMING NONPARAMETRIC BAYESIAN ANALYSES OF ORDERED BINOMIAL DATA

*William W. McDonald*

Approved for public release; distribution is unlimited.

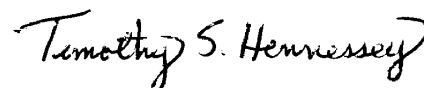


REPORT DOCUMENTATION PAGE					Form Approved OMB No. 0704-0188	
<p>The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding the burden estimate or any other aspect of this collection of information, including suggestion for reducing this burden, to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.</p> <p>PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.</p>						
1. REPORT DATE (DD-MM-YYYY)		2. REPORT TYPE		3. DATES COVERED (From - To)		
2 May 2003		Final Report				
4. TITLE AND SUBTITLE  MBR—A COMPUTER PROGRAM FOR PERFORMING NONPARAMETRIC BAYESIAN ANALYSES OF ORDERED BINOMIAL DATA				5a. CONTRACT NUMBER		
				5b. GRANT NUMBER		
				5c. PROGRAM ELEMENT NUMBER 0603782N		
6. AUTHOR(S)  William W. McDonald				5d. PROJECT NUMBER N0001499WX301410		
				5e. TASK NUMBER		
				5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Indian Head Division Naval Surface Warfare Center Indian Head, MD 20640-5035					8. PERFORMING ORGANIZATION REPORT NUMBER IHTR 2323	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Office of Naval Research Arlington VA 22217-5000					10. SPONSOR/MONITOR'S ACRONYM(S) ONR	
					11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12a. DISTRIBUTION/AVAILABILITY STATEMENT  Approved for public release; distribution is unlimited.						
13. SUPPLEMENTARY NOTES						
14. ABSTRACT  The MBR computer program computes posterior marginal distributions for binomial response probabilities associated with a set of $M$ -ordered stresses or stimuli. Exact solutions are achieved of the posterior marginal distribution functions, first published by Disch, which was based on Ramsey's $M$ -variate ordered Dirichlet joint prior. MBR assumes a related joint prior that is a mixture of Dirichlet distributions to obtain a class capable of representing arbitrary and quite general forms. The joint prior distribution is reconstructed from three percentile curves, such as the 10th, 50th, and 90th percentiles, of the prior marginal distributions as assigned by an expert. The code then calculates the posterior marginal distributions (mixtures of beta distributions) and constructs new percentile curves that reflect the effect of the data upon the priors in accordance with Bayes's law. Rapid and exact solutions are obtained by means of a recursive theory developed by the author.						
15. SUBJECT TERMS  Bayesian, Stimulus-response, Regression, Computer program, Nonparametric, Dirichlet, Disch						
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT  SAR	18. NUMBER OF PAGES  36	19a. NAME OF RESPONSIBLE PERSON Susan Simpson	
a. REPORT U	b. ABSTRACT U	c. THIS PAGE U			19b. TELEPHONE NUMBER (Include area code) (301) 744-4284	

This page intentionally left blank.

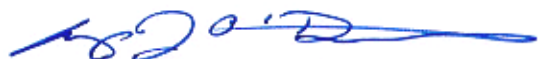
## FOREWORD

The present MBR code, written in the Mathcad (version 2001) language, was developed at the Indian Head Division, Naval Surface Warfare Center (NSWC). It represents an extensive revision and reformulation of an earlier code that was written in Fortran by the author and Mr. Patrick O'Neal at NSWC (White Oak Laboratory) circa 1988. Both codes were based on a theory developed by the author during the period of 1982 through 1984, which showed how arbitrary prior distributions for ordered response data could be represented by a mixture of ordered Dirichlet distributions and described how rapid and exact calculations of the posterior marginal distributions could be achieved by means of recursive relationships. A report on the theory has been recently published as a companion to this report and is included among the references. Over the years, the MBR program has been used by the Navy in a number of significant applications concerning the vulnerability of Naval structures to explosions. A digital copy of the code can be obtained by contacting Mr. Hennessey at [hennesseyts@ih.navy.mil](mailto:hennesseyts@ih.navy.mil) or sending a request to the Warheads Branch.



Timothy S. Hennessey  
Manager, Warheads Branch

Approved and released by:



Amy J. O'Donnell  
Director, Underwater Division

Released by:



Marc S. Magdinec  
Head, Weapons Department

This page intentionally left blank.

## CONTENTS

<i>Heading</i>	<i>Page</i>
Foreword.....	iii
Introduction.....	1
MBR Computer Program (Fully Expanded).....	3
Introduction to MBR.....	3
Input and Representation of The Prior Percentile Curves .....	3
Input of Observational and Nonobservational Data.....	8
Data Checking and Preparation of Dirichlet Prior.....	12
Calculation of The Posterior Marginal Distributions.....	14
Notes—Current Code is Version 5 .....	22
Concluding Comments.....	23
References.....	24

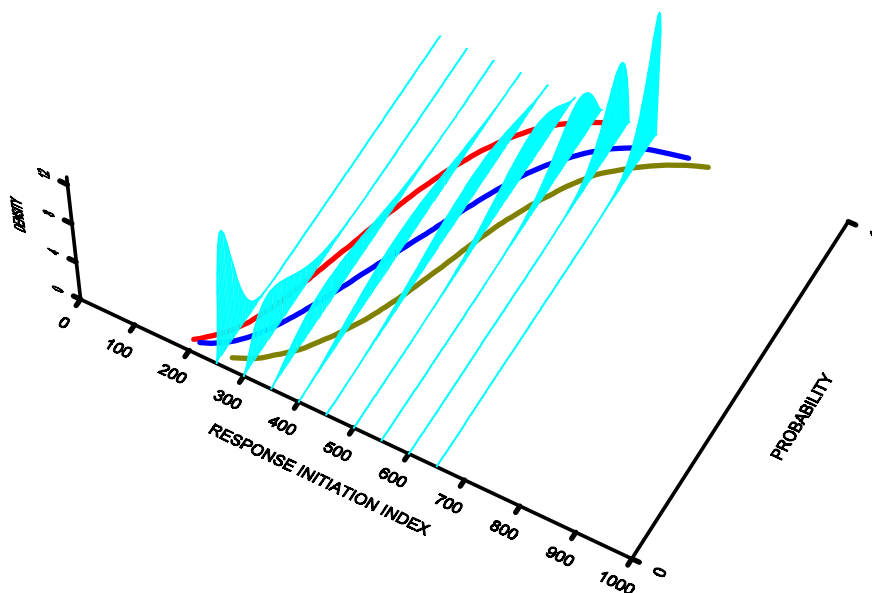
## Figures

1. Prior Marginal Densities and Percentile Curves.....	1
--	---

This page intentionally left blank.

## INTRODUCTION

The MBR (Monotone Bayesian Regression) computer program computes posterior marginal distributions for binomial response probabilities associated with a set of ordered stimulus levels. Usually these are values of a physics-based measure of the test environment severity. We shall call it a response initiation index function or, alternatively, a generalized stress. Use is made of posterior marginal distribution functions for Ramsey's  $M$ -variate ordered Dirichlet joint prior (Ramsey, 1972), which were first published by Disch (1981). Because these functions involve multiple nested summations, they are exceedingly difficult to calculate directly. MBR makes use of recursive expressions developed by the author (McDonald, 2003) that make it possible to achieve rapid and exact evaluations. Moreover, a joint prior consisting of a single  $M$ -variate ordered Dirichlet distribution is not sufficiently general to apply to most problems of interest. Consequently, MBR employs a related joint prior that is a mixture of  $M$ -variate ordered Dirichlet distributions to obtain a class capable of representing quite arbitrary and general forms. The mixed prior is constructed by having an expert provide a set of three percentile curves (including the median) that are functions of the initiation index as shown in Figure 1. Usually the curves are chosen as the 10th, 50th, and 90th percentiles of the prior marginal distributions. The code then assigns parameters to the mixed Dirichlet prior to match the percentiles and calculates the posterior marginal distributions using the response data in accordance with Bayes's law. An earlier Fortran version of the code was written in 1988 by the author and Mr. Patrick O'Neill at the Naval Surface Warfare Center (NSWC) (White Oak, MD). The current Mathcad code was written at the Indian Head Division, NSWC and completed in August of 1999.



**Figure 1. Prior Marginal Densities and Percentile Curves**



The MBR code (version 5) is listed in the following section and contains both text portions and mathematical coding. The code has an interactive format that instructs the code user to provide information concerning his or her prior percentile curves, to input the test data, and to set program constants that control the execution. It processes the data, provides a graphical display of the posterior distribution percentile curves, and calculates the 80% coverage interval for a 90% probability of failure.

The present code is written in the Mathcad (version 2001) language. Mathcad was chosen because it combines a user-friendly textbook-like style with very powerful mathematical algorithms, programming, and graphical capabilities. Execution proceeds from left to right and from top to bottom.

Previous Mathcad versions of the code were two-dimensional Mathcad work sheets with the interactive narrative and results running down the left-hand pages of the worksheet and the supporting equations spread out to the right. This two-dimensional structure made the earlier Mathcad codes unsuitable for publication as reports. In an effort to remedy this problem, MBR version 5 is organized vertically in the left-hand pages only and includes four areas of supporting code that may be hidden from view to enhance program readability or expanded to show programming details. The program with the detailed coding areas hidden is best for users whose primary goal is to analyze data. The areas are expanded in this report to show all elements of the program. The hideable areas are bounded by horizontal lines. The areas may be hidden or unhidden by double clicking on one of the lines. Generally, the code within a hideable area is directly relevant to the section that follows. The function of the coding within each area is indicated by a numbered *AREA* title written in 8-point font above or below the horizontal line, such as

↓ *AREA 1. INPUT & REPRESENTATION OF THE PRIOR PERCENTILE CURVES (CODE)*

---

which is found at the top of page 4. The texts and titles within the collapsible areas are presented in italics to make them distinguishable from the main text. Each collapsible area can also be locked and password protected.

## MBR COMPUTER PROGRAM (FULLY EXPANDED)

### Introduction to MBR

MBR is a computer program that is used to express and update (from test data) the probability  $p$  of some arbitrarily defined binary response as a function of an appropriately chosen quantity, which we symbolize by  $Y$ .  $Y$  is a function of the test conditions and is referred to as a response initiation index or a generalized stress. The model requires that  $p$  depend uniquely upon  $Y$  and that  $p$  increase (or be nondecreasing) with  $Y$ . No other assumption concerning their functional relationship is made. The statistical approach is Bayesian. As  $p$  is unknown for all values of  $Y$  (excepting possibly at zero and infinity), it is regarded as a random variable. The distribution of  $p$  at a given value of  $Y$  indicates the uncertainty of the response under the conditions implied by  $Y$ . Prior to the examination of test data, MBR requires descriptions of the distribution of  $p$  as a function of  $Y$  over the range of  $Y$  values of interest. These are provided in the form of three distribution percentile curves. MBR then constructs the joint prior distribution and combines it with the binomial test data via Bayes theorem to obtain posterior marginal distributions and updated percentile curves that indicate how the initial prior uncertainties should be revised in light of the test results.

### Input and Representation of the Prior Percentile Curves

MBR reconstructs prior marginal distributions from three probability percentile curves — the 50th percentile, and lower and upper percentiles chosen by the user. Enter now the values of the lower and upper percentiles:

$$Lower := 10 \qquad Upper := 90$$

To input the curves from files, skip to *Tabular Entry of Prior* in Area 2. To draw the curves graphically, indicate in the following matrix the  $Y$  values where each curve crosses the  $p$  values indicated by the column headings. BotLine and TopLine percentile values are arbitrarily set. Zero indicates lower threshold values.

$$BotLine := .1 \qquad TopLine := .9$$

$$GraphInput := \begin{pmatrix} \%ile Curve \backslash & Zero & BotLine & .5 & TopLine \\ Upper & 0 & 1.9 & 3 & 4.5 \\ 50 & 0 & 2.8 & 4 & 5.6 \\ Lower & 0 & 3.8 & 5 & 6.6 \end{pmatrix}$$

### Code for Input and Representation of the Prior Percentile Curves

#### Special Functions $Xcal$ & $Pcal$ and $XBurr$ & $PBurr$

$$\begin{aligned}
 Xcal(p, C, i) := & \left| \begin{array}{l}
 x_0 \leftarrow C_{1,0} \\
 a \leftarrow C_{i,1} \\
 b \leftarrow C_{i,2} \\
 \gamma \leftarrow C_{i,3} \\
 wm \leftarrow C_{i,4} \\
 pr \leftarrow \frac{p}{1-p} \\
 z \leftarrow \frac{1}{1 + pr^{-\gamma}} \\
 ex \leftarrow \frac{1}{a + b \cdot z} \\
 exln \leftarrow ex \cdot \ln(pr) \\
 x \leftarrow 10^{37} \text{ if } exln > 85 \\
 x \leftarrow x_0 \text{ if } exln < -85 \\
 x \leftarrow x_0 + wm \cdot pr^{ex} \text{ otherwise}
 \end{array} \right.
 \end{aligned}$$

$$\begin{aligned}
 Pcal(p, pmi, C, i) := & \left| \begin{array}{l}
 a \leftarrow C_{i,0} \\
 b \leftarrow C_{i,1} \\
 \gamma \leftarrow C_{i,2} \\
 pr \leftarrow \frac{1-p}{p} \\
 pmr \leftarrow \frac{1-pmi}{pmi} \\
 ps \leftarrow \frac{1}{\left( a + \frac{b}{1 + pmr \cdot pr^{\gamma}} \right)}
 \end{array} \right.
 \end{aligned}$$

```

XBurr( $ps, A, n$ ) :=
   $pr_B \leftarrow \frac{1 - ps_0}{ps_0}$ 
   $pr_T \leftarrow \frac{1 - ps_2}{ps_2}$ 
   $gl \leftarrow -\ln(pr_B)$ 
   $gu \leftarrow -\ln(pr_T)$ 
  for  $i \in 1..n$ 
     $x_{0_i} \leftarrow A_{i,0}$ 
     $wl \leftarrow A_{i,1} - A_{i,0}$ 
     $wm \leftarrow A_{i,2} - A_{i,0}$ 
     $wu \leftarrow A_{i,3} - A_{i,0}$ 
     $ru \leftarrow \frac{wm}{wu}$ 
     $rl \leftarrow \frac{wm}{wl}$ 
     $yl \leftarrow -\ln(rl)$ 
     $yu \leftarrow -\ln(ru)$ 
     $\gamma \leftarrow .9$ 
     $aa \leftarrow -1$ 
     $bb \leftarrow -1$ 
     $count \leftarrow 0$ 
    while  $[(aa < 0) + (aa + bb < 0)] \neq 0$ 
       $count \leftarrow count + 1$ 
      return "count = 20" if  $count = 20$ 
       $\gamma \leftarrow \gamma + .1$ 
       $zl \leftarrow \frac{1}{1 + pr_B^\gamma}$ 
       $zu \leftarrow \frac{1}{1 + pr_T^\gamma}$ 
       $bb \leftarrow \frac{\frac{gu}{yu} - \frac{gl}{yl}}{zu - zl}$ 
       $aa \leftarrow \frac{gu}{yu} - bb \cdot zu$ 
     $a_i \leftarrow aa$ 
     $b_i \leftarrow bb$ 
     $\gamma v_i \leftarrow \gamma$ 
     $wmv_i \leftarrow wm$ 
  augment( $x_0, augment(a, augment(b, augment(\gamma v, wmv))))$ 

```

```

PBurr( $ps, A, n$ ) :=
   $w_L \leftarrow \frac{1 - ps_0}{ps_0}$ 
   $w_U \leftarrow \frac{1 - ps_2}{ps_2}$ 
   $y_L \leftarrow -\ln(w_L)$ 
   $y_U \leftarrow -\ln(w_U)$ 
  for  $i \in 1..n$ 
     $zm \leftarrow \frac{1 - A_{i,1}}{A_{i,1}}$ 
     $h_L \leftarrow -\ln\left(\frac{1 - A_{i,0}}{A_{i,0} \cdot zm}\right)$ 
     $h_U \leftarrow -\ln\left(\frac{1 - A_{i,2}}{A_{i,2} \cdot zm}\right)$ 
     $Dif \leftarrow \frac{h_U}{y_U} - \frac{h_L}{y_L}$ 
     $\gamma \leftarrow .9$ 
     $aa \leftarrow -1$ 
     $bb \leftarrow 1$ 
     $count \leftarrow 0$ 
    while  $aa \leq bb \cdot 0.09984$ 
       $count \leftarrow count + 1$ 
      return "count = 20" if  $count = 20$ 
       $\gamma \leftarrow \gamma + .1$ 
       $wTerm \leftarrow \frac{1}{1 + w_L^\gamma}$ 
       $bb \leftarrow \frac{Dif}{\frac{1}{1 + w_U^\gamma} - wTerm}$ 
       $aa \leftarrow \frac{h_L}{y_L} - bb \cdot wTerm$ 
     $a_i \leftarrow aa$ 
     $b_i \leftarrow bb$ 
     $\gamma v_i \leftarrow \gamma$ 
  ( $a$   $b$   $\gamma v$ )
  augment( $a$ , augment( $b$ ,  $\gamma v$ ))

```

## Tabular Entry of Prior

Code to create *GraphInput* array from files rather than by the user should be inserted here.

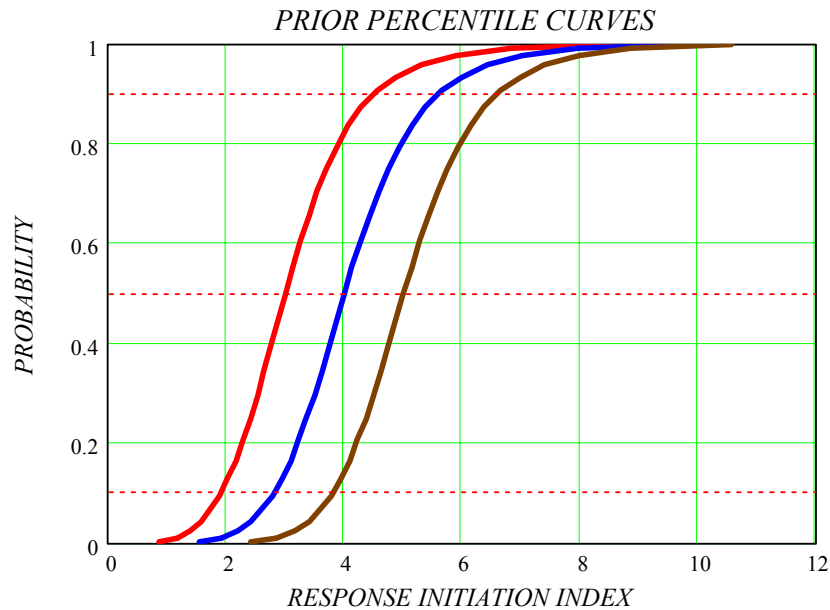
## GraphInput Processing and Display of Prior Percentile Curves

```

Zero ≡ "Zero"
nplot := 30    j := 1..nplot-1    pvj := .5 * (1 - cos( (j * π) / nplot ))
ps := submatrix(GraphInput, 0, 0, 2, 4)T
A := submatrix(GraphInput, 0, 3, 1, 4)
XC := XBurr(ps, A, 3)
pU := pv    p50 := pv    pL := pv
xUj := Xcal(pUj, XC, 1)    x50j := Xcal(p50j, XC, 2)    xLj := Xcal(pLj, XC, 3)
curveU := augment(xU, pU)
curve50 := augment(x50, p50)
curveL := augment(xL, pL)
nU := rows(curveU) - 1    n50 := rows(curve50) - 1    nL := rows(curveU) - 1
jU := 1..nU    j50 := 1..nU    jL := 1..nL
Range := ceil(max(curveL) + 1)    Range = 12
kk := 0..1    xx0 := 0    xx1 := Range

```

## Display of Input Prior Percentile Curves



## Input of Observational and Nonobservational Data

Test results are provided to MBR through a four-column matrix called “TestData.” The first column is the test number (order is arbitrary), the second column is the value of the test, the third column is the number of tests conducted at that level, and the fourth column is the number of responses observed. Enter information into the TestData matrix now.

	TestData			
	"Test Number"	"Res.Init.Index"	"#Tests"	"#Responses"
	1	1	1	0
	2	2	1	0
	3	3	1	0
	4	4	1	0
	5	5	1	1
	6	6	1	1
	7	7	1	0
	8	8	1	0
	9	9	1	1
	10	10	1	1
	11	11	1	1
	12	12	1	1
	13	13	1	1
	14	14	1	1

## Nonobservational Data

In addition to calculating the posterior percentiles at the values of  $Y$  listed in the TestData matrix, MBR will also calculate the posterior percentiles at an arbitrary number of nonobservational values evenly distributed across the response initiation index range. These values are useful both for predictions and display of the posterior percentile curves.

**Input the number of nonobservational values desired:**  $n_{nonobs} := 5$

The Data matrix, augmented by nonobservational values and placed in order of ascending  $Y$  values, is shown below. (Note: Data of any duplicated  $Y$  entries are combined.)

⇓ *AREA 2. DATA MATRIX SUBMITTED FOR PROCESSING (CODE)*

### Code for Processing Data Matrix

#### Range for Insertion of Nonobservational Data

$$X_{.05} := Xcal(.05, XC, 2) \quad X_{.95} := Xcal(.95, XC, 2)$$

$$X_{.05} = 2.476 \quad X_{.95} = 6.269$$

```

ObsData := | return 0 if  $n_{nonobs} = 0$ 
               $x_{int} \leftarrow \frac{X_{.95} - X_{.05}}{n_{nonobs} + 1}$ 
               $Mat_{0,0} \leftarrow \text{"nonobs"}$ 
               $Mat_{0,1} \leftarrow X_{.05} + x_{int}$ 
               $Mat_{0,2} \leftarrow 0$ 
               $Mat_{0,3} \leftarrow 0$ 
              return Mat if  $n_{nonobs} = 1$ 
              for  $k \in 1..n_{nonobs} - 1$ 
                   $Mat_{k,0} \leftarrow \text{"nonobs"}$ 
                   $Mat_{k,1} \leftarrow Mat_{k-1,1} + x_{int}$ 
                   $Mat_{k,2} \leftarrow 0$ 
                   $Mat_{k,3} \leftarrow 0$ 
              Mat

```



```

Data := | return TestData if ObsData = 0
        | Stack ← stack(TestData, ObsData)
        | ndat ← rows(Stack) − 1
        | D ← submatrix(Stack, 1, ndat, 0, 3)
        | D ← csort(D, 1)
        | nn ← 0
        | for k ∈ 0..ndat − 1
        |   | if Dnn,1 = Dk,1
        |   |   | Dnn,0 ← Dk,0 if Dnn,0 = "obs"
        |   |   | Dnn,2 ← Dnn,2 + Dk,2
        |   |   | Dnn,3 ← Dnn,3 + Dk,3
        |   | otherwise
        |   |   | nn ← nn + 1
        |   |   | Dnn,0 ← Dk,0
        |   |   | Dnn,1 ← Dk,1
        |   |   | Dnn,2 ← Dk,2
        |   |   | Dnn,3 ← Dk,3
        | stack[( "Test Number" "Res.Init.Index" "#Tests" "#Responses" ), D]

```

(*M* is number of response initiation indices in Data matrix.)

$M := \text{rows}(\text{Data}) - 1$     $M = 19$     $i := 1..M$     $Y := \text{Data}^{\langle 1 \rangle}$     $TOL := 10^{-10}$

Given    $Xcal(p, XC, k) = u$     $p > 0$     $p < 1$     $Xfun(p, XC, k, u) := \text{Find}(p)$

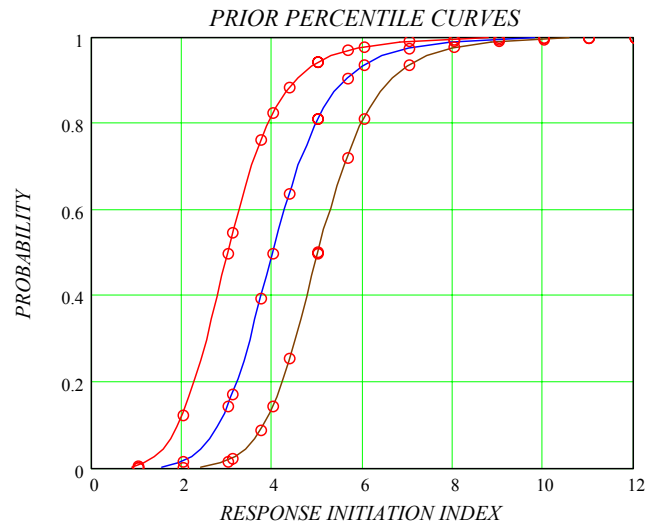
$p := 0.5$

$p_{lo_i} := Xfun(p, XC, 3, Y_i)$

$p_{mid_i} := Xfun(p, XC, 2, Y_i)$

$p_{hi_i} := Xfun(p, XC, 1, Y_i)$

Graph below shows  $p_{lo}$ ,  $p_{mid}$ , and  $p_{hi}$  values calculated from percentile curves at the test and nonobservational  $Y$  values.



|| AREA 2. DATA MATRIX SUBMITTED FOR PROCESSING (CODE)

### Data Matrix Submitted for Processing

*Data =*

	0	1	2	3
0	"Test Number"	"Res.Init.Index"	"#Tests"	"#Responses"
1	1	1	2	0
2	2	2	1	0
3	3	3	1	0
4	"nonobs"	3.108	0	0
5	"nonobs"	3.74	0	0
6	4	4	1	0
7	"nonobs"	4.372	0	0
8	5	5	1	1
9	"nonobs"	5.004	0	0
10	"nonobs"	5.637	0	0
11	6	6	1	1
12	7	7	1	0
13	8	8	1	0
14	9	9	1	1
15	10	10	1	1

## Data Checking and Preparation of Dirichlet Prior

### Prior Distribution Accuracy

MBR approximates the prior marginal distributions as mixtures of  $J$  Beta distributions and the joint prior distribution as a mixture of  $J$  ordered Dirichlet distributions. The number of terms in the mixtures,  $J$ , affects the accuracy of these representations. Usually,  $J$  set in the range of 20 to 50, gives sufficient accuracy for most problems.

**Set J now:**  $J := 20$

⇓ AREA 3. DATA CHECKING AND PREPARATION OF DIRICHLET PRIOR

---

### Code for Data Checking and Preparation of Dirichlet Prior

*MBR now checks that ordering constraints imposed on the prior mixture distributions are satisfied.*

*Locations of mixed distribution kernels are determined by equating their means to unevenly spaced fractiles of the reconstructed prior marginals. The spacing is determined by a beta distribution that is symmetric about  $p = 0.5$ . The parameter  $\kappa$  determines the degree of concentration of points toward the tails of the distributions. This fitting technique performs better than the previous method of equating the modes to the fractiles of equally spaced probabilities. The value of  $\beta$  is chosen according to the number of kernels  $J$  so that twice the kernel's standard deviation equals  $1/J$  of the span.*

$$p_{curves} := \left( \frac{Lower}{100} \quad .5 \quad \frac{Upper}{100} \right)^T \quad p_{curves} = \begin{pmatrix} 0.1 \\ 0.5 \\ 0.9 \end{pmatrix}$$

$$PA := augment(p_{lo}, augment(p_{mid}, p_{hi}))$$

$$PC := PBurr(p_{curves}, PA, M)$$

$$j := 1..J$$

$$\sigma_{mult} := 1 \quad \sigma_{\beta} := \left( \frac{1}{2 \cdot J} \right) \cdot \sigma_{mult} \quad \sigma_{\beta} = 0.025$$

$$\beta := \left( \frac{J}{\sigma_{mult}} \right)^2 - 3 \quad \beta = 397$$

$$\kappa := 0.1$$

$$p_{ord_j} := pbeta\left(\frac{j}{J+1}, 1+\kappa, 1+\kappa\right)$$

$$\mu_{star_{j,i}} := Pcal(p_{ord_j}, p_{mid_i}, PC, i)$$

$$\Phi_j := \frac{1}{J} \quad a_{i,j} := (\beta + 2) \cdot \mu_{star_{j,i}}$$

$$b_{i,j} := \beta + 2 - a_{i,j}$$

$$p_{star_{j,i}} := \frac{a_{i,j} - 1}{\beta} \quad p_{star_{j,M+1}} := 1$$

$$OrderingCheck := \left| \begin{array}{l} \text{for } j \in 1..J \\ \quad \text{for } i \in 2..M \\ \quad \quad \text{return ("Error (i,j) = " } i \text{ } j \text{ ) if } \mu_{star_{j,i-1}} > \mu_{star_{j,i}} \\ \quad \quad \text{"OK"} \end{array} \right.$$

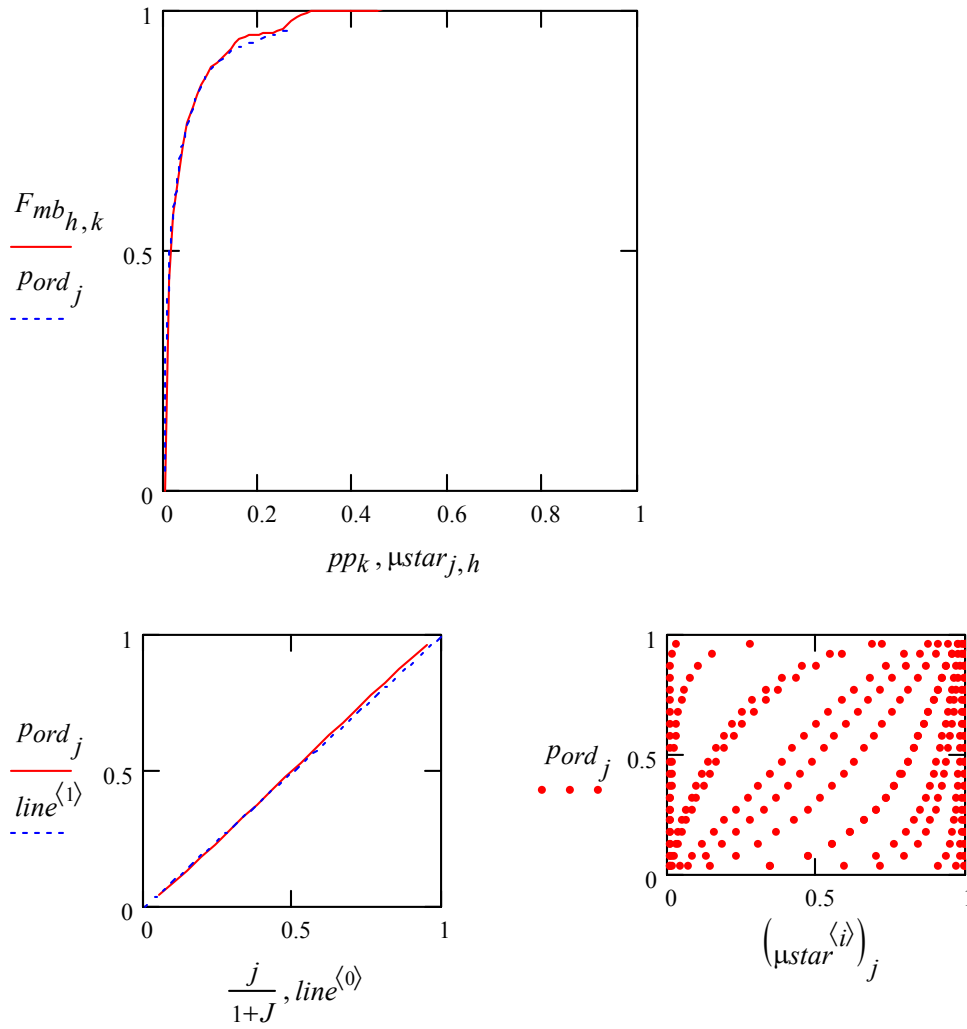
**Compare Dirichlet Marginal (Mixed Beta) at  $\Upsilon_h$  with Input  $\mu$ Star Values**

$$pMixedBeta(i, p, a, b, \Phi, J) := \sum_{j'=1}^J \Phi_{j'} \cdot pbeta(p, a_{i,j'}, b_{i,j'})$$

$$dMixedBeta(i, p, a, b, \Phi, J) := \sum_{j'=1}^J \Phi_{j'} \cdot dbeta(p, a_{i,j'}, b_{i,j'})$$

$$line := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad nk := 100 \quad k := 0..nk \quad pp_k := \frac{k}{nk} \quad \sigma_{mult} = 1 \quad J = 20$$

$$\text{Set: } h := 2 \quad F_{mb_{h,k}} := pMixedBeta(h, pp_k, a, b, \Phi, J)$$




---

↑ AREA 3. DATA CHECKING AND PREPARATION OF DIRICHLET PRIOR

*OrderingCheck* = "OK"

If *OrderingCheck* is not "OK", try reducing the value of  $J$ .

### Calculation of the Posterior Marginal Distributions

The percentiles of the posterior marginal distribution functions are now calculated by the MBR and are plotted below alongside the prior distribution function percentile curves.

Calculation time for this example problem takes approximately 30 seconds on an Intel 1200-MHz computer.

---

⇓ AREA 4. CALCULATION OF POSTERIOR MARGINAL DISTRIBUTIONS AND DISPLAY OF POSTERIOR PERCENTILE CURVES

---

**Code for Calculation of Posterior Marginal Distributions and Display of Posterior Percentile Curves**

**Posterior Parameters:**

$$M = 19$$

$$nf_i := Data_{i,3}$$

$$nt_i := Data_{i,2}$$

$$ns_i := nt_i - nf_i$$

$$nt_{max} := \max(nt) \quad nt_{max} = 2$$

$$mapp_0 := 0 \quad mapp_i := \text{if}[i = 1, 1, mapp_{i-1} + (i-2) \cdot nt_{max} + 1]$$

$$mapq_0 := 0 \quad mapq_i := \text{if}[i = 1, 1, mapq_{i-1} + (M - i + 1) \cdot nt_{max} + 1]$$

$$nfl_{e_i} := \sum_{k=1}^i nf_k$$

$$ns_{ge_i} := \sum_{k=i}^M ns_k$$

$$ns_{le_i} := \sum_{k=1}^i ns_k$$

$$nf_{ge_i} := \sum_{k=i}^M nf_k$$

```

 $C_p := \text{for } j \in 1..J$ 
     $C_{p,j,mapp_1} \leftarrow 1$ 
     $\text{break if } M < 2$ 
     $T \leftarrow 1$ 
     $\text{for } l \in 1..nfl_{e_1} \quad \text{if } nfl_{e_1} \geq 1$ 
         $T \leftarrow T \cdot \frac{(a_{1,j+l-1})}{(a_{2,j+l-1})}$ 
     $\text{for } k \in 0..nsl_{e_1}$ 
         $C_{p,j,mapp_2+k} \leftarrow T \cdot \text{combin}(ns_1, k)$ 
         $nk1 \leftarrow nfl_{e_1} + k$ 
         $T \leftarrow T \cdot \frac{(a_{1,j+nk1})}{(a_{2,j+nk1})}$ 
     $\text{break if } M < 3$ 
     $\text{for } i \in 3..M$ 
         $T \leftarrow 1$ 
         $\text{for } l \in 1..nfl_{e_{i-1}} \quad \text{if } nfl_{e_{i-1}} \geq 1$ 
             $T \leftarrow T \cdot \frac{(a_{i-1,j+l-1})}{(a_{i,j+l-1})}$ 
         $\text{for } k \in 0..nsl_{e_{i-1}}$ 
             $sum \leftarrow 0$ 
             $A1 \leftarrow (k - nsl_{e_{i-1}})$ 
             $A2 \leftarrow (k - nsl_{e_{i-2}})$ 
             $\text{for } r \in \max(A1)..\min(A2)$ 
                 $sum \leftarrow sum + \text{combin}(nsl_{e_{i-1}}, k - r) \cdot C_{p,j,mapp_{i-1}+r}$ 
             $C_{p,j,mapp_i+k} \leftarrow T \cdot sum$ 
             $nk1 \leftarrow nfl_{e_{i-1}} + k$ 
             $T \leftarrow T \cdot \frac{(a_{i-1,j+nk1})}{(a_{i,j+nk1})}$ 
     $C_p$ 

```

```

 $C_q := \text{for } j \in 1..J$ 
   $C_{qj, \text{map}q_M} \leftarrow 1$ 
   $\text{break if } M < 2$ 
   $T \leftarrow 1$ 
   $\text{for } l \in 1..ns_{ge_M} \quad \text{if } ns_{ge_M} \geq 1$ 
     $T \leftarrow T \cdot \frac{(b_{M,j+l-1})}{(b_{M-1,j+l-1})}$ 
   $\text{for } k \in 0..nf_{ge_M}$ 
     $C_{qj, \text{map}q_{(M-1)+k}} \leftarrow T \cdot \text{combin}(nf_M, k)$ 
     $nk \leftarrow ns_{ge_M} + k$ 
     $T \leftarrow T \cdot \frac{(b_{M,j+nk})}{(b_{M-1,j+nk})}$ 
   $\text{break if } M < 3$ 
   $\text{for } h \in 1..M-2$ 
     $i \leftarrow M - h - 1$ 
     $T \leftarrow 1$ 
     $\text{for } l \in 1..ns_{ge_{i+1}} \quad \text{if } ns_{ge_{i+1}} \geq 1$ 
       $T \leftarrow T \cdot \frac{(b_{i+1,j+l-1})}{(b_{i,j+l-1})}$ 
     $\text{for } k \in 0..nf_{ge_{i+1}}$ 
       $sum \leftarrow 0$ 
       $A1 \leftarrow (k - nf_{i+1} \ 0)$ 
       $A2 \leftarrow (k \ nf_{ge_{i+2}})$ 
       $\text{for } s \in \max(A1)..\min(A2)$ 
         $sum \leftarrow sum + \text{combin}(nf_{i+1}, k-s) \cdot C_{qj, \text{map}q_{i+1}+s}$ 
       $C_{qj, \text{map}q_i+k} \leftarrow T \cdot sum$ 
       $nk \leftarrow ns_{ge_{i+1}} + k$ 
       $T \leftarrow T \cdot \frac{(b_{i+1,j+nk})}{(b_{i,j+nk})}$ 
   $C_q$ 

```



### Posterior Distribution Function Calculation

```

UF(i, p) := SUM ← 0
for j ∈ 1..J
  Π ← 1
  c ← ai,j + bi,j
  r ← nf lei
  if r > 0
    Π ←  $\frac{a_{i,j}}{c}$ 
    for t ∈ 1..r-1      if r > 1
      Π ←  $\Pi \cdot \frac{(a_{i,j} + t)}{(c + t)}$ 
    c ← c + r
  s ← ns gei
  if s > 0
    Π ←  $\Pi \cdot \frac{b_{i,j}}{c}$ 
    for t ∈ 1..s-1      if s > 1
      Π ←  $\Pi \cdot \frac{(b_{i,j} + t)}{(c + t)}$ 
  T ← 1
  sum ← 0
  A ← ai,j + nf lei
  B ← bi,j + ns gei
  C ← A + B
  kmax ← if(i = 1, 0, ns lei-1)
  k'max ← if(i = M, 0, nf gei+1)
  for k ∈ 0..kmax
    T' ← 1
    C' ← C + k
    sum' ← 0
    for k' ∈ 0..k'max
      sum' ← sum' ...
        + (-1)k' · C qj, mapqi+k' · T' · pbeta(p, A + k, B + k')
      T' ← T' ·  $\frac{B + k'}{C' + k'}$ 
    sum ← sum + (-1)k · C pj, mappi+k · T · sum'
    T ← T ·  $\frac{A + k}{C + k}$ 
  SUM ← SUM + Φj · Π · sum
SUM

```

Note that the normalizing constant  $FNorm$  is the same regardless of the value of  $i$  chosen in  $UF(i, 1)$ . Disch (1981) regarded this as a check of code accuracy.

$$FNorm := UF(1, 1) \quad FNorm = 1.773 \times 10^{-4}$$

$$F(i, p) := \frac{UF(i, p)}{FNorm}$$

*Distribution Function inverse calculated by divide and conquer*

```

Finverse(ii, frac) :=
    win ← .0001
    pL ← 0
    pR ← 1
    p ←  $\frac{ii}{M}$ 
    count ← 0
    stop ← 0
    while stop = 0
        count ← count + 1
        return p if count > 100
        G ← F(ii, p)
        if G < (1 - win) · frac
            pL ← p
            p ←  $\frac{p_L + p_R}{2}$ 
        if G > (1 + win) · frac
            pR ← p
            p ←  $\frac{p_L + p_R}{2}$ 
        stop ← 1 otherwise
    p

```

$$r_{i,0} := Finverse(i, p_{curves_0})$$

$$r_{i,1} := Finverse(i, p_{curves_1})$$

$$r_{i,2} := Finverse(i, p_{curves_2})$$

$$u' := submatrix(Y, 1, M, 0, 0)$$

$$r' := submatrix(r, 1, M, 0, 2)$$

$$vlo := cspline(u', r^{\langle 0 \rangle})$$

$$v50 := cspline(u', r^{\langle 1 \rangle})$$

$$vup := cspline(u', r^{\langle 2 \rangle})$$

$$k := 0..50$$

$$u_k := \frac{k}{50} \cdot \max(u')$$

$$j' := 0..M - 1$$

	0	1	2
0	0	0	0
1	0	0	$3.71431049547697 \cdot 10^{-6}$
2	$3.19376244748894 \cdot 10^{-10}$	$7.95063219572368 \cdot 10^{-4}$	0.012669613486842
3	$4.14276123046875 \cdot 10^{-3}$	0.026810495476974	0.129060444078947
4	$6.10753109580592 \cdot 10^{-3}$	0.034796463815789	0.155530427631579
5	0.03620750025699	0.128045333059211	0.368215460526316
6	0.064978348581414	0.196443256578947	0.471808182565789
7	0.133983812834087	0.325336657072368	0.613898026315789
8	0.334189967105263	0.574022795024671	0.793778268914474
9	0.3359105963456	0.575709292763158	0.794793379934211
10	0.582815872995477	0.769980982730263	0.89464689555921
11	0.703163548519737	0.842336554276316	0.927477384868421
12	0.894301565069901	0.942192479183799	0.971880461040296
13	0.959650541606702	0.976931120219984	0.988926937705592
14	0.982188375372636	0.991033453690378	0.996671174701891
15	0.990755583110609	0.996504131116365	0.999245091488487
16	0.99473089920847	0.998739945261102	0.999904532181589
17	0.996898450349507	0.99964061536287	0.999996260592812
18	0.998219540244655	0.999938262136359	0.999999982059786
19	0.99906325340271	0.999995857477188	0.99999999997101

$$plo_k := \text{interp}(vlo, u', r^{\langle 0 \rangle}, u_k)$$

$$p50_k := \text{interp}(v50, u', r^{\langle 1 \rangle}, u_k)$$

$$pup_k := \text{interp}(vup, u', r^{\langle 2 \rangle}, u_k)$$

$$uu := \frac{u'_0 + u'_{M-1}}{2}$$

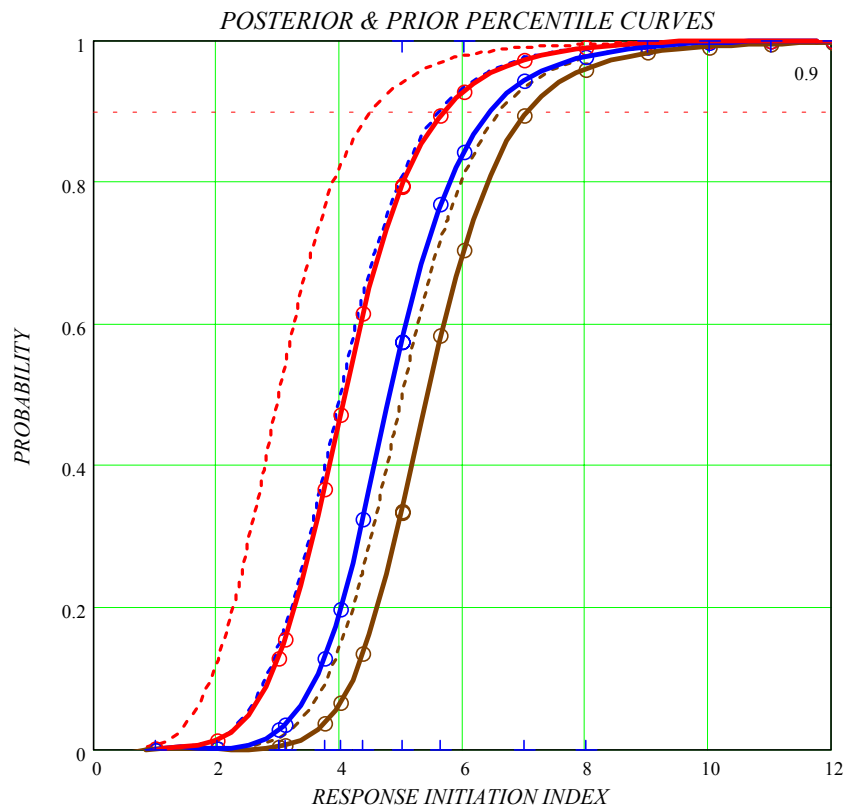
$$u_L := \text{root}(\text{interp}(vup, u', r^{\langle 2 \rangle}, uu) - .9, uu) \quad u_L = 5.688$$

$$u_R := \text{root}(\text{interp}(vlo, u', r^{\langle 0 \rangle}, uu) - .9, uu) \quad u_R = 7.053$$

---

↑ AREA 4. CALCULATION OF POSTERIOR MARGINAL DISTRIBUTIONS AND DISPLAY OF POSTERIOR PERCENTILE CURVES

## Posterior Percentile Curves



**80 Percent Coverage Interval**

The 80% coverage interval  $(u_L, u_R)$  for the response index associated with a 0.9 probability of response is

$$u_L = 5.68 \quad u_R = 7.05$$

**Notes—Current Code is Version 5**

MBR3: Version 3 differs from Version 2 in the following respects: Version 3 calculates the posterior density function along with the distribution function and computes the distribution function percentiles by a Newton-Raphson scheme. (Version 2 used a divide-and-conquer technique.) Plots of the posterior density function and distribution function are available. (Look below the FNorm calculation.)

MBR4: Version 4 goes back to the divide-and-conquer scheme for calculating the inverse posterior distribution function. Although faster, the Newton-Raphson algorithm was found to be less robust. Posterior density function coding used in Version 3 remains.

MBR5: Version 5 is the code of this document. It differs from Version 4 only in the use of a vertical format and hideable, lockable areas.

## CONCLUDING COMMENTS

MBR appears to be functioning properly. Future improvements contemplated include development of a more efficient algorithm for calculating the inverse posterior distribution as currently performed by Finverse. Possible choices include a hybrid Newton-Raphson and divide-and-conquer scheme.

The PBurr and XBurr functions calculate the distribution function and inverse, respectively, of a modified form of the Burr distribution. The Burr distribution can be found discussed in Kendall and Stuart (1960). Representations were needed in MBR of a distribution function and its inverse whose parameters were the median and two percentiles. These are used for graphing purposes and, most importantly, to construct modified Burr representations of the prior marginals from the percentile curves, which are then used to assign parameters to the mixed beta forms of the prior marginals and the mixed Dirichlet joint prior. Difficulties may be encountered with the modified Burr distributions when the prior percentile curves are unusually or improperly positioned relative to each other. Hence, these too may be the subject of future improvements.

## REFERENCES

- Disch, D. (1981), "Bayesian Nonparametric Inference for Effective Doses in a Quantal Response Experiment," *Biometrics* 37, 713-722.
- Ramsey, F.L. (1972), "A Bayesian Approach to Bioassay," *Biometrics* 28, 841-58.
- McDonald, W. (1984), "A Bayesian Model for the Analysis of Quantal Response Data," draft and extension of 1982 paper presented at the Virginia Polytechnic and State University symposium *Reflections on Bayesian Approaches in Operations Research, Probability, and Statistics*, Blacksburg, VA.
- McDonald, W. (2003), *A Bayesian Model for the Analysis of Quantal Response Data*, Indian Head Division, Naval Surface Warfare Center, IHTR 2503 (publication of the 1984 draft), 5 May 2003.
- Kendall, M., and Stuart, A. (1969), *The Advanced Theory of Statistics*, Vol. 1, 3rd Edition, Griffin & Company, Ltd., London.

## DISTRIBUTION

OFFICE OF NAVAL RESEARCH ATTN CODE 32 (B ALMQUIST) 800 N QUINCY STREET ARLINGTON VA 22217-5660	1	DEPT OF MANAGEMENT SCIENCE ATTN R SOYER GEORGE WASHINGTON UNIVERSITY 2121 EYE ST NW WASHINGTON DC 20052	1
OFFICE OF NAVAL RESEARCH ATTN CODE 32 (T. SWEAN) 800 N QUINCY STREET ARLINGTON VA 22217-5660	1	INST OF STAT & DECISION SCIENCE ATTN J BERGER DUKE UNIVERSITY BOX 90251 DURHAM NC 27708-0251	1
ADVANCED TECH & RESEARCH CORP ATTN J GOELLER SPRINGPOINTE EXECUTIVE CENTER 15210 DINO DRIVE BURTONSVILLE MD 20866-1172	1	THE FUQUA SCHOOL OF BUSINESS ATTN R WINKLER DUKE UNIVERSITY BOX 90251 DURHAM NC 27708-0251	1
ADVANCED TECH & RESEARCH CORP ATTN C MCCLURE SPRINGPOINTE EXECUTIVE CENTER 15210 DINO DRIVE BURTONSVILLE MD 20866-1172	1	STATISTICS DEPARTMENT ATTN F RAMSEY OREGON STATE UNIVERSITY CORVALIS OR 97331-4606	1
SRI, INTERNATIONAL ATTN AH236 X6093 (P. GEFKEN) 333 RAVENSWOOD AVENUE MENLO PARK CA 94025	1	R EASTERLING 51 AVENIDA DEL SOL CEDAR CREST NM 87008	1
DEPT OF ENGINEERING MGMT & SYS ENG ATTN T MAZZUCHI GEORGE WASHINGTON UNIVERSITY 2121 EYE ST NW WASHINGTON DC 20052	1	NOVELLUS SYSTEMS INC ATTN Y CUI 4000 N FIRST STREET SAN JOSE CA 95134	1
STATISTICS DEPARTMENT ATTN N SINGPURWALLA GEORGE WASHINGTON UNIVERSITY 2121 EYE ST NW WASHINGTON DC 20052	1	JHU/CPIA ATTN SECURITY OFFICER 10630 LITTLE PATUXENT PKWY STE 202 COLUMBIA MD 21044-3200	1 CD
STATISTICS DEPARTMENT ATTN S BOSE GEORGE WASHINGTON UNIVERSITY 2121 EYE ST NW WASHINGTON DC 20052	1	K KIM DIRECTOR DEFENSE SPECIAL WEAPONS AGENCY 6801 TELEGRAPH RD ALEXANDRIA VA 22310-3398	1



TECHNICAL LIBRARY DIRECTOR DEFENSE SPECIAL WEAPONS AGENCY 6801 TELEGRAPH RD ALEXANDRIA BA 22310-3398	1	TECHNICAL LIBRARY LOS ALAMOS TECHNICAL LABORATORY PO BOX 1663 LOS ALAMOS NM 87545	1
TECHNICAL LIBRARY NAVAL RESEARCH LABORATORY 4555 OVERLOOK DR SW WASHINGTON DC 20375-5320	1	TECHNICAL LIBRARY SANDIA NATIONAL LABORATORIES PO BOX 5800 ALBUQUERQUE NM 87185	1
TECHNICAL LIBRARY CARDEROCK DIVISION NAVAL SURFACE WARFARE CENTER BETHESDA MD 20084-5000	1	TECHNICAL LIBRARY SANDIA NATIONAL LABORATORIES PO BOX 969 LIVERMORE CA 94551-0969	1
TECHNICAL LIBRARY DAHLGREN DIVISION NAVAL SURFACE WARFARE CENTER DAHLGREN VA 22448-5000	1	TECHNICAL LIBRARY LAWRENCE LIVERMORE NATIONAL LAB PO BOX 808 LIVERMORE CA 94550-0427	1
TECHNICAL LIBRARY SUPERINTENDENT NAVAL POST GRADUATE SCHOOL MONTEREY CA 93940	1	TECHNICAL LIBRARY CENTER FOR NAVAL ANALYSES 4401 FORD AVENUE PO BOX 16268 ALEXANDRIA VA 22302-0268	1
TECHNICAL LIBRARY US ARMY RESEARCH LABORATORY APG MD 21005-5068	1	TECHNICAL LIBRARY INSTITUTE FOR DEFENSE ANALYSIS 1801 N BEAUREGARD ST. ALEXANDRIA VA 22311	1
S. WILKERSON US ARMY RESEARCH LABORATORY APG MD 21005-5068	1	D NICHOLSON UNIVERSITY OF CENTRAL FLORIDA MECHANICAL ENGINEERING AND AEROSPACE SCIENCES DEPARTMENT PO BOX 25000 ORLANDO FL 32816-0450	1
LIBRARY PENNSYLVANIA STATE UNIVERSITY APPLIED RESEARCH LABORATORY PO BOX 30 STATE COLLEGE PA 16804-0030	1	D. HARVILLE DEPT OF STATISTICS IOWA STATE UNIVERSITY AMES IA 50011	1
TECHNICAL LIBRARY THE JOHNS HOPKINS UNIVERSITY APPLIED PHYSICS LABORATORY JOHNS HOPKINS ROAD LAUREL MD 20723-6099	1		

J GIOVANOLA  
LABORATOIRE DE CONCEPTION DES  
SYSTEMES MECHANQUES DGM  
ECOLE POLYTECHNIQUE FEDERAL  
DE LAUSANNE EUBLENS  
1015 LAUSANNE  
SWITZERLAND 1

W MARTINEZ  
OFFICE OF NAVAL RESEARCH  
BALLSTON TOWER I  
800 NORTH QUINCY STREET  
ARLINGTON VA 22217-5660 1

**Internal:**

ED	1
ST	1
40	1
430C (L TAYLOR)	1
4330 (T HENNESSEY)	1
440 (K BURROWS)	1
4430 (S LANDSBERG)	1
4430Q (D MCINTOSH)	1

**Electronic Copy:**

ADMINISTRATOR  
DEFENSE TECH INFORMATION CTR  
ATTN JACK RIKE OCA  
8725 JOHN J KINGMAN RD STE 0944  
FT BELVOIR VA 22060-6218

This page intentionally left blank.

